Pseudo Dirichlet sets and a new cardinal invariant (General and Geometric Topology)

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Pseudo Dirichlet sets and a new cardinal invariant

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Abstract

Z. Bukovská [5] proved that $\mathbf{p} \leq \mathrm{non}(\mathcal{PD})$, where $\mathcal{PD}$ denotes the set of all pseudo-Dirichlet sets. In this paper, we shall show that $\mathbf{p}$ can be replaced by $\mathbf{h}$ in this inequality. It is known that $\mathbf{p} < \mathbf{h}$ is consisitent (see [1]). So, the equality $\mathbf{p} = \mathrm{non}(\mathcal{PD})$ can not be proved. This is a partial answer of problem 2 in [6]. Next, we shall introduce a certain cardinal invariant $\mathbf{f}$ and show that $\mathrm{add}(\mathcal{N}) \leq \mathbf{f} \leq \mathrm{non}(\mathcal{PD})$. Also, we shall construct two generic models such that one satisfies the inequality $\mathbf{b} < \mathbf{f}$ and another satisfies the inequality $\mathbf{f} < \mathrm{non}(\mathcal{PD})$.

1 Introduction

Throughout this paper, we shall use the standard terminologies for forcing of set theory and cardinal invariants on ω (see [3]). For each $a \in \mathbf{R}$, we denote by $\|a\|$ the distance of a and the set of integers $\mathbf{Z}$. Let A be a subset of the unit interval $[0, 1]$. A is called a *pseudo Dirichlet set*, if there exists an $X \in [\omega]^\omega$ such that

$$\forall a \in A \, \forall^\infty n \in X \, (\, \|na\| < \frac{1}{|X \cap n| + 1} \,).$$

We denote the set of all pseudo Dirichlet sets by $\mathcal{PD}$. Z. Bukovská [5] showed that $\mathbf{p} \leq \mathrm{non}(\mathcal{PD})$. Let $\mathbf{h}$ be the least cardinal κ such that the boolean algebra $\mathcal{P}(\omega)/\mathrm{fin}$ does not satisfy the κ-distributive law.

Theorem 1.1 $\quad \mathbf{h} \leq \mathrm{non}(\mathcal{PD})$.

Proof For each $a \in [0, 1)$, let $\|a\|^*$ denote the unique real number r such that $0 \leq r < 1$ and $a = r \pmod{\mathbf{Z}}$. To show this theorem, let $A \subset [0, 1]$ and $|A| < \mathbf{h}$.

For each $a \in A$, take a maximal almost disjoint set $W_a \subset [\omega]^\omega$ such that

$$\forall n \in X\, \forall m \in X \setminus n\ (\ |\|na\|^* - \|ma\|^*| < \frac{1}{|X \cap n| + 1}\ ),\ \text{for all } X \in W_a.$$

Since $|A| < \mathbf{h}$, there exists a maximal almost disjoint set W such that W is an refinement of all W_a's. Take a $Y \in W$. Choose some $Y' = \{\, y_i \mid i < \omega \,\}_< \in [Y]^\omega$ such that

$$|Y \cap [y_i, y_{i+1})| \geq i \text{ and } y_{i+1} - y_i < y_{i+2} - y_{i+1},\ \text{for all } i < \omega.$$

Let $Z = \{\, y_{i+1} - y_i \mid i < \omega \,\}$. We complete the proof by showing that

$$\forall^\infty n \in Z\ (\ \|na\| < \frac{1}{|Z \cap n| + 1}\ ),\ \text{for all } a \in A.$$

Let $a \in A$. Since W is a refinement of W_a, there exists an $X \in W_a$ such that $Y \subset^* X$. Take an $i < \omega$ such that $Y \setminus y_i \subset X$. Then, for any $j \in [i+1, \omega)$, it holds that

$$\|(y_{j+1} - y_j)a\| \leq |\|y_{j+1}a\|^* - \|y_j a\|^*| \leq \frac{1}{|X \cap y_j| + 1} < \frac{1}{j+1}. \qquad \square$$

2 Combinatorial principle wIn_2

T. Bartoszynski [2] introduced the notion of slalom and, using this, investigated systematically the relations between combinatorics and cardinal invariants which are associated by the null ideal $\mathcal{N}$ and the meager ideal $\mathcal{M}$. The following statement In_2 and the theorem are some of them.

Definition 2.1 *For $h \in {}^\omega\omega$ and $F \subset {}^\omega\omega$, define the statement $\mathrm{In}_2(F, h)$ by*

$$\mathrm{In}_2(F, h) \equiv \exists \varphi \in \prod_{n<\omega} [\omega]^{\leq h(n)}\ \forall f \in F\, \forall^\infty n < \omega\ (\ f(n) \in \varphi(n)\ ).$$

The statement $\mathrm{In}_2(F, \mathrm{id}_\omega)$ is denoted by $\mathrm{In}_2(F)$, where id_ω is the identity function on ω.

Theorem 2.1 (Bartoszynski [2]) $\mathrm{add}(\mathcal{N}) = \min\{\, |F| \mid F \subset {}^\omega\omega$ *and not* $\mathrm{In}_2(F)\,\}$.

In this section, we shall introduce the statement wIn_2 which is some variant of In_2. And we shall study relations between wIn_2 and $\mathrm{non}(\mathcal{PD})$.

Definition 2.2 *For $H,\ h \in {}^{\omega}\omega$ and $F \subset \prod_{n<\omega} H(n)$, define the statement* $\mathrm{wIn}_2(F, h, H)$ *by*

$$\mathrm{wIn}_2(F,h,H) \equiv \exists\, \varphi \in \prod_{n<\omega} [H(n)]^{\leq h(n)}\ \forall f \in F\ \forall^{\infty} n < \omega\ (\ f(n) \in \varphi(n)\ ).$$

$\mathrm{wIn}_2(F, H)$ *denotes the statement* $\mathrm{wIn}_2(F, \mathrm{id}_{\omega}, H)$. *Let*

$$\mathbf{f} = \min\{\, |F| \mid \exists\, H \in {}^{\omega}\omega\ (\ F \subset \prod_{n<\omega} H(n)\ \textit{and not}\ \mathrm{wIn}_2(F,H)\ )\,\}.$$

The following lemma can be easily proved by the result of Bartorszynski.

Lemma 2.2 $\mathrm{add}(\mathcal{N}) = \min\{\, \mathbf{b}, \mathbf{f}\,\}$. □

The main result of this section is the following theorem.

Theorem 2.3 $\mathbf{f} \leq \mathrm{non}(\mathcal{PD})$.

To show this theorem, we need some notations and lemmas.

A sequence $\langle\, I_n \mid n < \omega\,\rangle$ is called an *interval partition* of ω, if there exists an increasing function $f \in {}^{\omega}\omega$ such that $f(0) = 0$ and, for all $n < \omega$, $I_n = \{\, k < \omega \mid f(n) \leq k < f(n+1)\,\}$.

The next lemma can be deduced from [6, Proposition 1]. But, for a convenience for the reader, we give a proof.

Lemma 2.4 *Let $n < \omega$ and $0 < m,\ k < \omega$. Then, there exists some $p < \omega$ such that*

$$\forall\, a_0, \cdots, a_{m-1} \in [0,1]\ \exists\, s < \omega\ (\ n \leq s < p\ \textit{and}\ \forall\, i < m\ (\ \|sa_i\| < \frac{1}{k}\ )\ ).$$

Proof By induction on $1 \leq m < \omega$.

Case 1. $m = 1$.

We claim thta $p = nk + 1$ satisfies the condition. To show this, let $a \in [0,1]$. Define the mapping $\sigma : X = \{\, nj \mid j = 1, \cdots, k\,\} \to k$ by, for each $j < k$,

$$\sigma(nj) = \text{“ the unique } i \text{ such that } \frac{i}{k} \leq \|nja\|^{*} < \frac{i+1}{k}\text{ ”}.$$

If there exists some nj such that $\sigma(nj) = 0$ or $k-1$, then $s = nj$ is a required one.

Otherwise, there exist $i < j \leq k$ such that $\sigma(ni) = \sigma(nj)$ and $s = n(j-i)$ is a required one.

Case 2. $m = m' + 1$.

By induction hypothesis, there exist $0 = p_0 < p_1 < \cdots < p_k$ such that

$$\forall a_0, \cdots, a_{m'-1} \in [0,1]\, \exists s < \omega \,(\, p_j + n \leq s < p_{j+1} \text{ and } \forall i < m' \,(\, \|sa_i\| < \frac{1}{2k} \,)\,),$$

for $j < k$.

We show that $p = p_k$ satisfies the condition. So, let $a_0, \cdots, a_{m'} \in [0,1]$. By the choise of p_j (for $j < k$), there exist $s_0, \cdots, s_{k-1} < \omega$ such that

$$p_j + n \leq s_j < p_{j+1} \text{ and } \forall i < m' \,(\, \|s_j a_i\| < \frac{1}{2k} \,)\,), \text{ for } j < k.$$

Then, it holds that

$$\|s_j a_{m'}\| < \frac{1}{k}, \text{ for some } j < k$$

or

$$\|s_j a_{m'} - s_{j'} a_{m'}\| < \frac{1}{k}, \text{ for some } j < j' < k.$$

In either cases, similar to case 1, we can take a required element s. $\square$

Corollary 2.5 *There is an interval partition $\langle I_n \mid n < \omega \rangle$ which satisfies*

$$(*)\begin{cases} \text{For any } n < \omega \text{ and } a_0, \cdots, a_{n-1} \in [0,1), \text{ there exists some } k \in I_n \text{ such that} \\ \qquad \|ka_i\| < 2^{-n}, \text{ for all } i < n. \end{cases}$$

$\square$

Proof of Thorem 2.3 Take an interval partition $\langle I_n \mid n < \omega \rangle$ which satisfies (*) in the previous corollary. Define $H \in {}^\omega\omega$ by

$$H(n) = 2^n \sum_{k \leq n} |I_k|, \text{ for all } n < \omega.$$

To show the theorem, let $A \subset [0,1]$ and $|A| < \mathbf{f}$. For each $a \in A$, define $f_a \in \prod_{n<\omega} H(n)$ by

$$\frac{f_a(n)}{H(n)} \leq a < \frac{f_a(n)+1}{H(n)}, \text{ for all } n < \omega.$$

Since $|A| < \mathbf{f}$, there exists a $\varphi \in \prod_{n<\omega} [H(n)]^n$ such that

$$\forall a \in A \, \forall^\infty n < \omega \; (\; f_a(n) \in \varphi(n) \;).$$

For each $n < \omega$, take $s_n \in I_n$ such that

$$\|s_n \frac{j}{H(n)}\| < 2^{-n}, \text{ for all } j \in \varphi(n).$$

We complete the proof by showing that

$$\forall a \in A \, \forall^\infty n < \omega \; (\; \|s_n a\| < 2^{-n+1} \;).$$

So, let $a \in A$. Take an $m < \omega$ such that

$$\forall n \geq m \; (\; f_a(n) \in \varphi(n) \;).$$

Then, for any $n \geq m$, since $\dfrac{f_a(n)}{H(n)} \leq a < \dfrac{f_a(n)+1}{H(n)}$, it holds that

$$s_n \frac{f_a(n)}{H(n)} \leq s_n a < s_n \frac{f_a(n)+1}{H(n)}.$$

So,

$$\|s_n a\| \leq \|s_n \frac{f_a(n)}{H(n)}\| + \frac{s_n}{H(n)} < 2^{-n+1}.$$

□

Note that what we really proved is $\min\{ |F| \mid \text{not } \mathrm{wIn}_2(F, H) \} \leq \mathrm{non}(\mathcal{PD})$, where H is a function defined in the proof of Theorem 2.3.

3 The cardinal invariant f

In the previous section, we introduced the cardinal invariant $\mathbf{f}$ and showed the equality $\mathrm{add}(\mathcal{N}) = \min\{ \mathbf{b}, \mathbf{f} \}$. Both of $\mathrm{add}(\mathcal{N})$ and $\mathbf{b}$ appear in the Cihoń's diagram. It seems to be an interesting problem to check the relations between $\mathbf{f}$ and other cardinals in the diagram. Since it is known that $\mathcal{PD} \subset \mathcal{N} \cap \mathcal{M}$, it holds that $\mathbf{f} \leq \min\{ \mathrm{non}(\mathcal{N}), \mathrm{non}(\mathcal{M}) \}$. So, $\mathbf{f}$ seems to be not so large. If the inequality $\mathbf{f} \leq \mathbf{b}$ always holds, then $\mathbf{f}$ is equal to $\mathrm{add}(\mathcal{N})$ and $\mathbf{f}$ does not become a new cardinal invariant. In this section, we shall show that there exists a generic model which satisfies the inequality $\mathbf{b} < \mathbf{f}$.

Definition 3.1 *For each $H \in {}^\omega\omega$, define the forcing notion $Q(H)$ by*

$$Q(H) = \{ p \in \prod_{n<\omega} [H(n)]^{\leq n} \mid \exists k < \omega \, \forall n < \omega \; (\; |p(n)| \leq k \;) \},$$

$$q \leq p \quad \textit{iff} \quad \forall\, n < \omega\ (\ p(n) \subset q(n)\ ).$$

Define $\tau_H : Q(H) \to \omega$ *by*

$$\tau_H(p) = \min\{\ k < \omega \mid \forall\, n < \omega\ (\ |p(n)| \leq k\ )\ \}.$$

Using the density argument, the following lemma can be proved easily.

Lemma 3.1 *Let* $H \in {}^\omega\omega$ *and* $\mathcal{G}$ *be* V*-generic on* $Q(H)$*. In* $V[\mathcal{G}]$*, define* $\varphi \in \prod_{n<\omega} \mathcal{P}(H(n))$ *by*

$$\varphi(n) = \bigcup\{\ p(n) \mid p \in \mathcal{G}\ \}.$$

Then, it holds that

(1) $|\varphi(n)| \leq n$, *for all* $n < \omega$,

(2) $\forall\, g \in (\prod_{n<\omega} H(n))^V\ \forall^\infty\, n < \omega\ (\ g(n) \in \varphi(n)\ ).$ □

Lemma 3.2 $Q(H)$ *satisfies the* ω_1*-chain condition.*

Proof Let $W \subset Q(H)$ and $|W| = \omega_1$. Replace W by a certain subset of W, if necessary, we can assume that, for some $k < \omega$,

$$\tau_H(p) = k \text{ and } p{\restriction}2k = p'{\restriction}2k, \text{ for all } p,\, p' \in W.$$

Then, every elements of W are mutually compatible. □

Lemma 3.3 *Every unbounded family of functions in* ${}^\omega\omega \cap V$ *is still unbounded in* $V^{Q(H)}$.

Bartszinski and Judah [3, Theorem 6.4.13] proved that any finite support iteration by forcing notions which preserved the unboundedness in ${}^\omega\omega$ does not add a dominating function. So, starting a ground model which satisfies CH, by choosing appropreate H's, we can construct an ω_2-stage finite support iteration P such that V^P satisfies $\mathbf{b} = \omega_1$ and $\mathbf{f} = \omega_2$.

In order to prove Lemma 3.3, we need a result of Brendle and Judah [4]. Let P be a forcing notion which satisfies the ω_1-chain condition and $\tau : P \to \omega$ be a homomorphism. Following Brendle and Judah [4], we say that (P, τ) is *nice*, if it satisfies

$$\left\{ \begin{array}{l} \text{For any predence set } \{ p_i \mid i < \omega \} \subset P, \text{ it holds that} \\ \qquad \forall m < \omega \, \exists n < \omega \, \forall q \in P \text{ (if } \tau(q) \leq m, \text{ then } \exists i < n \text{ (} q \upharpoonleft p_i \text{)).} \end{array} \right.$$

Theorem 3.4 (Brendle and Judah [4]) *Let (P, τ) be a nice forcing notion. Then, every unbounded family of functions in ${}^{\omega}\omega \cap V$ is still unbounded in $V^{Q(H)}$.* □

Proof of Lemma 3.3 It suffices to show that $(Q(H), \tau_H)$ is nice. So, let $\{ p_i \mid i < \omega \}$ be a predence subset of $Q(H)$ and $m < \omega$. To get a contradiction, assume that, for each $n < \omega$, there exists a condition $q_n \in Q(H)$ such that

$$\tau_H(q_n) \leq m \text{ and } \forall i < n \text{ (} q_n \perp p_i \text{).}$$

Since $\{ q_n \restriction k \mid n < \omega \}$ is a finite set for every $k < \omega$, we can choose $X_k \in [\omega]^\omega$ by induction on $k < \omega$ such that

$$X_{k+1} \subset X_k \text{ and } \forall n, n' \in X_k \text{ (} q_n \restriction (k+1) = q_{n'} \restriction (k+1) \text{).}$$

Define $r \in Q(H)$ by

$$r(k) = q_n(k), \text{ for some/all } n \in X_k.$$

Note that $\tau_H(r) \leq m$. Since $\{ p_i \mid i < \omega \}$ is predence, there exists $i < \omega$ such that r is compatible with p_i. Let $k = \tau_H(p_i) + m$. Take $n \in X_k$ such that $i < n$. Since $i < n$, it holds that p_i and q_n are incompatible. Since $\tau_H(p_i) + \tau_H(q_n) \leq k$, it holds that $\exists j < k$ ($|p_i(j) \cup q_n(j)| > j$). By this and the fact that $r \restriction k = q_n \restriction k$, r is incompatible with p_i. This is a contradiction. □

4 Consistency of $\mathbf{f} < \mathrm{non}(\mathcal{PD})$

Concerning about the cardinal invariant associated by In_2, T. Bartoszynski [2] pointed out implicitly that, if two functions h_0, $h_1 \in {}^{\omega}\omega$ satisfies that

$$\lim_{n<\omega} h_i(n) = \infty, \text{ for } i = 0, 1,$$

then $\min\{ |F| \mid \text{ not } \mathrm{In}_2(F, h_0) \} = \min\{ |F| \mid \text{ not } \mathrm{In}_2(F, h_1) \}$.

In this section, we shall show that, for any $H \in {}^{\omega}\omega$, $\mathbf{f}$ may not be equal to $\min\{ |F| \mid \text{ not } \mathrm{wIn}_2(F, H) \}$. Using this, we shall prove the consistency of $\mathbf{f} <$

non($\mathcal{PD}$). Henceforce, $H \in {}^\omega\omega$ is an arbitrary, but fixed function on ω. For each $k < \omega$, let

$$T_k \; (= T_k^H) = \{\, q \in Q(H) \mid \tau_H(q) \leq k \,\}.$$

Define $H_0,\ H_1 : \omega \times \omega \to \omega$ by

$$H_0(k,m) = \min\left\{\, l < \omega \;\middle|\; \begin{array}{c} \forall\, \delta : l \to [\omega_2]^{\leq k}\ \exists\, S \in [\,l\,]^m \exists\, v \in [\omega_2]^{\leq k} \\ \forall\, i,\, j \in S \text{ (if } i \neq j, \text{ then } \delta(i) \cap \delta(j) = v \text{)} \end{array} \right\},$$

$$H_1(k,m) = \min\left\{\, l < \omega \;\middle|\; \begin{array}{c} \forall\, \delta : l \to T_k\ \exists\, S \in [\,l\,]^m \exists\, q \in Q(H) \\ \forall\, i \in S \text{ (} q \leq \delta(i) \text{)} \end{array} \right\}.$$

Note that H_0 is a recursive function. And, H_1 is an H-recursive function, since it holds that

$$\exists\, q' \in Q(H)\ \forall\, q \in S \text{ (} q' \leq q \text{)} \quad \text{iff} \quad \forall\, i < mk \text{ (} |\bigcup_{q \in S} q(i)| \leq i \text{)}\ , \text{ for any}$$

$S \in [T_k]^m$.

Define $H_2,\ H^* : \omega \to \omega$ by

$$H_2(k) = \underbrace{H_1(k, H_1(k, H_1(\cdots, H_1}_{k \text{ times}}(k, k+1)\cdots))),$$

$$H^*(k) = H_0(k, H_2(k)).$$

Define an ω_2-stage finite support iteration P_α (for $\alpha \leq \omega_2$) associated with $\dot{Q}_\alpha$ (for $\alpha < \omega_2$) by

$$\Vdash_\alpha \dot{Q}_\alpha = Q(H)\ , \quad \text{for all } \alpha < \omega_2.$$

Let $P(H) = P_{\omega_2}$. It holds that

$$V^{P(H)} \models \forall\, F \subset \prod_{n<\omega} H(n) \text{ (if } |F| \leq \omega_1, \text{ then } \mathrm{wIn}_2(F, H) \text{)}.$$

The purpose of this section is to show

Theorem 4.1 $\quad V^{P(H)} \models$ *not* $\mathrm{wIn}_2((\prod_{n<\omega} H^*(n))^V, H^*)$.

Corollary 4.2 *Suppose that* $V \models$ CH. *Let* $H \in {}^\omega\omega$ *be the function which is defined in the proof of Theorem 2.3. Then, it holds that*

$$V^{P(H)} \models \mathbf{f} = \omega_1 \textit{ and } \mathrm{non}(\mathcal{PD}) = \omega_2. \qquad \square$$

To show Theorem 4.1, we need some definitions and lemmas. Let

$$D = \{ p \in P(H) \mid \forall \alpha \in \mathrm{supp}(p) \, (\, p{\restriction}\alpha \text{ decides } \tau_H(p(\alpha)) \,) \, \}.$$

The following lemma can be proved easily.

Lemma 4.3 *D is dense in $P(H)$.* □

Define $\rho : D \to \omega$ by

$$\rho(p) = \min \left\{ k < \omega \;\middle|\; \begin{array}{l} \quad |\mathrm{supp}(p)| \leq k \\ \text{and} \\ \forall \alpha \in \mathrm{supp}(p) \, (\, p{\restriction}\alpha \Vdash_\alpha \tau_H(p(\alpha)) \leq k \,) \end{array} \right\}.$$

For each 各 $k < \omega$, let

$$D_k = \{ p \in D \mid \rho(p) \leq k \, \}.$$

Lemma 4.4 *Let $k < \omega$ and $\delta : H^*(k) \to D_k$. Then, there exist $p^+ \in P(H)$ and $P(H)$-name $\dot{S}$ which satisfy* (1), (2).

(1) $\Vdash \dot{S} \subset H^*(k)$ *and* $|\dot{S}| \geq k+1$.

(2) $\forall i < H^*(k) \, \forall p' \leq p^+$ (*if* $p' \Vdash i \in \dot{S}$, *then* $p' \leq \delta(i)$).

Proof Let $k < \omega$. Define l_m (for $m \leq k$) by

$$\begin{cases} l_0 & = \quad k+1 \\ l_{m+1} & = \quad H_1(k, l_m) \end{cases}.$$

Note that $H^*(k) = H_0(k, l_k)$. Assume that $\delta : H^*(k) \to D_k$. Since $\langle \, \mathrm{supp}(\delta(i)) \mid i < H^*(k) \, \rangle : H^*(k) \to [\omega_2]^{\leq k}$, by the choise of H_0, there exist $S_0 \in [H^*(k)]^{l_k}$ and $v \in [\omega_2]^{\leq k}$ such that

$$\forall i, j \in S_0 \, (\text{ if } i \neq j, \text{ then } \mathrm{supp}(\delta(i)) \cap \mathrm{supp}(\delta(j)) = v \,).$$

Define $p \in P(H)$ by

$$\mathrm{supp}(p) = \bigcup \{ \, \mathrm{supp}(\delta(i)) \mid i \in S_0 \, \} \setminus v,$$

$$p(\alpha) = \delta(i)(\alpha), \text{ if } \alpha \in \mathrm{supp}(\delta(i)) \text{ and } i \in S_0.$$

Let $n = |v|$ and $v = \{ \, \alpha_1, \cdots, \alpha_n \, \}_<$. Note that $n \leq k$. By induction on $1 \leq m \leq n$, choose P_{α_m}-names $\dot{S}_m$, $\dot{q}_m$ such that

(3) $\Vdash_{\alpha_m} \dot{S}_m \in [\dot{S}_{m-1}]^{l_{k-m}}$ and $\dot{q}_m \in \dot{Q}_{\alpha_m}$,

(4) $p{\restriction}\alpha_m \cup \langle \, \dot{q}_j \mid 1 \leq j < m \, \rangle \Vdash_{\alpha_m} \dot{q}_m \leq \delta(i)(\alpha_m)$, for all $i \in \dot{S}_m$.

We must show that these can be chosen. Assume that $m \leq n$ and $\dot{S}_j$, $\dot{q}_j$ were chosen, for $j < m$. Since H_1 is abusolute and $H_1(k, l_{k-m}) = l_{k-m+1}$, it holds that

$p \restriction \alpha_m \cup \langle \dot{q}_j \mid 1 \leq j < m \rangle \Vdash_{\alpha_m} \exists q \in Q(H)\ \exists S \in [\dot{S}_{m-1}]^{l_{k-m}}\ \forall i \in S\ (\ q \leq \delta(i)(\alpha_m)\ )$.

Using this, it can be possible to choose $\dot{S}_m$, and $\dot{q}_m$.

Let $p^+ = p \cup \langle \dot{q}_m \mid 1 \leq m \leq n \rangle$, $\dot{S} = \dot{S}_n$. It is clear that this p^+ and $\dot{S}$ satisfy (1) in the lemma. In order to show that these satisfy (2), assume that

$$i < H^*(k) \text{ and } p' \leq p^+ \text{ and } p' \Vdash i \in \dot{S}.$$

Since $\Vdash_P \dot{S} = \dot{S}_n \subset \dot{S}_{n-1} \subset \cdots \subset S_0$, $i \in S_0$. For each $m = 1, \cdots, n$, since $\dot{S}_m$ is a P_{α_m}-name, it holds that $p' \restriction \alpha_m \Vdash_{\alpha_m} i \in \dot{S}_m$. By this, since $p' \leq p^+$, we have that

$$p' \restriction \alpha_m \Vdash_{\alpha_m} \dot{q}_m \leq \delta(i)(\alpha_m)\ , \text{ for all } m = 1, \cdots, n.$$

So, $p' \leq \delta(i)$. □

Lemma 4.5 *Let $k < \omega$. Assume that a $P(H)$-name $\dot{a}$ satisfies*

$$\Vdash \dot{a} \in [H^*(k)]^{\leq k}.$$

Then, there exists some $j < H^*(k)$ such that

$$\forall p \in D_k\ (\ \textit{not}\ p \Vdash j \in \dot{a}\ ).$$

Proof Suppose not. Take $\delta : H^*(k) \to D_k$ such that

$$\delta(j) \Vdash j \in \dot{a}, \text{ for all } j < H^*(k).$$

By the previous lemma, there exist $p^+ \in P(H)$ and $P(H)$-name $\dot{S}$ such that

$$\Vdash \dot{S} \subset H^*(k) \text{ and } |\dot{S}| \geq k+1,$$

$$\forall i < H^*(k)\ \forall p' \leq p^+\ (\text{ if } p' \Vdash i \in \dot{S}, \text{ then } p' \leq \delta(i)\ ).$$

Then, it holds that $p^+ \Vdash \dot{S} \subset \dot{a}$. This contradicts that $p^+ \Vdash |\dot{S}| \geq k+1$ and $|\dot{a}| \leq k$. □

Proof of Theorem 4.1 Assume that $\Vdash_{P(H)} \dot{\varphi} \in \prod_{k<\omega} [H^*(k)]^k$. Using the previous lemma, for each $k < \omega$, take a $j_k < H^*(k)$ such that

$$\forall p \in D_k\ (\text{ not } p \Vdash j_k \in \dot{\varphi}(k)\ ).$$

We claim that $\Vdash \exists^\infty k < \omega$ ($j_k \notin \dot\varphi(k)$). Suppose not. Then, there exist $p \in D$ and $n < \omega$ such that

$$p \Vdash \forall k > n \, (\, j_k \in \dot\varphi(k) \,).$$

Take $k > n$ such that $p \in D_k$. Then, it holds that $p \Vdash j_k \in \dot\varphi(k)$. But, this contradicts the choise of j_k. $\square$

Added in proof:

After the completion of this paper, Dr. Kada [7] have proved that $\mathbf{d} < \mathrm{non}(\mathcal{PD})$ is consistent with ZFC.

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